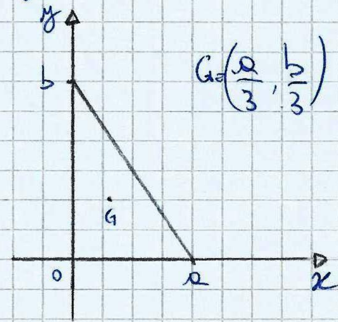
Fisica Matematica

Baricentri e matrici d'inerzia

Ricaviamo baricentri e matrici d'inerzia per alcune figure piane elementari (omogenee):

- triangolo rettangolo:



$y = -\frac{b}{a}x + b$

Baricentro:

$$G-O = \frac{1}{M}\int (P-O)\cdot\rho(P)\cdot d\tau(P) =$$
$$= \frac{1}{M}\iint \rho x\,dx\,dy \iint \rho y\,dx\,dy$$
$$x_G = \frac{\rho}{M}\int_0^a x\,dx\int_0^{-\frac{b}{a}x+b} dy =$$
$$= \frac{\rho}{M}\int_0^a\left(-\frac{b}{a}x^2+bx\right)dx =$$
$$= \frac{\rho}{M}\left[-\frac{b}{a}\frac{x^3}{3}+\frac{b}{2}x^2\right]_0^a$$

Notiamo che $\rho = \frac{M}{\frac{ab}{2}} = \frac{2M}{ab} \Rightarrow \frac{\rho}{M} = \frac{2}{ab}$;

$$x_G = \frac{2}{\cancel{ab}}\cdot\cancel{b}\left(-\frac{a^3}{3a}+\frac{a^2}{2}\right) = -\frac{2}{3}a + a = \frac{a}{3}$$

Analogamente si ottiene $y_G = \frac{b}{3}$:

$$y_G = \frac{\rho}{M}\int_0^b y\,dy\int_0^{-\frac{a}{b}y+a}dx = \frac{2}{ab}\left[-\frac{a}{b}\frac{y^3}{3}+\frac{a}{2}y^2\right]_0^b = \frac{b}{3}$$

Passiamo alle matrice d'inerzia rispetto all'origine O:

$$(I_O)_{11} = \iint\rho y^2\,dx\,dy = \rho\int_0^a dx\int_0^{-\frac{b}{a}x+b} y^2\,dy = \rho\frac{1}{3}\int_0^a \left(-\frac{b}{a}x+b\right)^3 dx =$$
$$= \frac{2M}{ab}\cdot\frac{1}{3}\left(-\frac{a}{4b}\right)\left[\left(-\frac{b}{a}x+b\right)^4\right]_0^a = +\frac{2M}{12b^2}\left(+b^4\right) = \frac{Mb^2}{6}$$

Analogamente si ottiene $(I_O)_{22} = \frac{Ma^2}{6}$:

$$(I_O)_{22} = \frac{2M}{ab}\int_0^b dy\int_0^{-\frac{a}{b}y+a}x^2\,dx = \frac{2M}{ab}\cdot\frac{1}{3}\cdot\left(-\frac{b}{4a}\right)\cdot\left[\left(-\frac{a}{b}y+a\right)^4\right]_0^b = \frac{Ma^2}{6}$$

Infine:

$$(I_O)_{1,2} = -\rho\iint xy\,dx\,dy = -\frac{2M}{ab}\int_0^a x\,dx\int_0^{-\frac{b}{a}x+b} y\,dy = -\frac{M}{ab}\int_0^a x\left(-\frac{b}{a}x+b\right)^2 dx =$$
$$= -\frac{M}{ab}\int_0^a\left(\frac{b^2}{a^2}x^3 - \frac{2b^2}{a}x^2 + b^2x\right)dx = -\frac{M}{ab}\left[\frac{b^2}{4a^2}x^4 - \frac{2b^2}{3a}x^3+\frac{b^2}{2}x^2\right]_0^a =$$
$$= -\frac{M}{ab}\left(\frac{b^2a^2}{4} - \frac{2b^2a^2}{3}+\frac{b^2a^2}{2}\right) = -Mab\left(\frac{1}{4}-\frac{2}{3}+\frac{1}{2}\right) =$$
$$= -Mab\,\frac{3-8+6}{12} = -\frac{Mab}{12}$$

$$(I_O)_{33} = (I_O)_{11}+(I_O)_{22} = \frac{M}{6}\left(a^2+b^2\right)$$

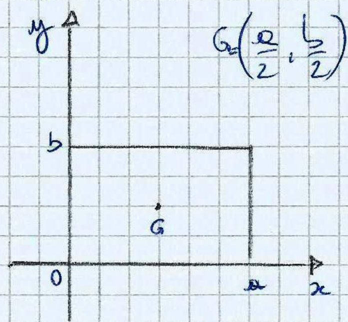
Essendo la figura piana, $(I_o)_{13} = (I_o)_{23} = 0$. La matrice I_o è:

$$I_o = \begin{pmatrix} \frac{Mb^2}{6} & -\frac{Mab}{12} & 0 \\ -\frac{Mab}{12} & \frac{Ma^2}{6} & 0 \\ 0 & 0 & \frac{M}{6}(a^2+b^2) \end{pmatrix}$$

Usando la formula di trasposizione calcoliamo I_G (matrice d'inerzia rispetto al baricentro):

$$I_G = \begin{pmatrix} \frac{Mb^2}{6} & -\frac{Mab}{12} & 0 \\ -\frac{Mab}{12} & \frac{Ma^2}{6} & 0 \\ 0 & 0 & \frac{M}{6}(a^2+b^2) \end{pmatrix} - M \begin{pmatrix} \frac{b^2}{9} & -\frac{ab}{9} & 0 \\ -\frac{ab}{9} & \frac{a^2}{9} & 0 \\ 0 & 0 & \frac{a^2+b^2}{9} \end{pmatrix} = \begin{pmatrix} \frac{Mb^2}{18} & +\frac{Mab}{36} & 0 \\ +\frac{Mab}{36} & \frac{Ma^2}{18} & 0 \\ 0 & 0 & \frac{M(a^2+b^2)}{18} \end{pmatrix}$$

- rettangolo:



Baricentro:

$$x_G = \frac{\rho}{M} \int_0^a x\, dx \int_0^b dy = \frac{1}{a\cancel{b}} \cdot \frac{a^2}{2} \cdot \cancel{b} = \frac{a}{2}$$

$$y_G = \frac{\rho}{M} \int_0^b y\, dy \int_0^a dx = \frac{1}{\cancel{a}b} \cdot \frac{b^2}{2} \cdot \cancel{a} = \frac{b}{2}$$

Entrate della matrice d'inerzia:

$$(I_o)_{11} = \iint \rho\, y^2 dx\, dy = \frac{M}{ab} \int_0^a dx \int_0^b y^2 dy = \frac{M}{\cancel{ab}} \cancel{a} \cdot \frac{b^{\cancel{3}\,2}}{3} = \frac{Mb^2}{3}$$

$$(I_o)_{22} = \iint \rho\, x^2 dx\, dy = \frac{M}{ab} \int_0^b dy \int_0^a x^2 dx = \frac{M}{\cancel{ab}} \cancel{b} \cdot \frac{a^{\cancel{3}\,2}}{3} = \frac{Ma^2}{3}$$

$$(I_o)_{12} = -\iint \rho\, xy\, dx\, dy = -\frac{M}{ab} \int_0^a x\, dx \int_0^b y\, dy = -\frac{M}{\cancel{ab}} \cdot \frac{a^{\cancel{2}}}{2} \cdot \frac{b^{\cancel{2}}}{2} = -M\frac{ab}{4}$$

$$(I_o)_{33} = (I_o)_{11} + (I_o)_{22} = \frac{M}{3}(a^2+b^2)$$

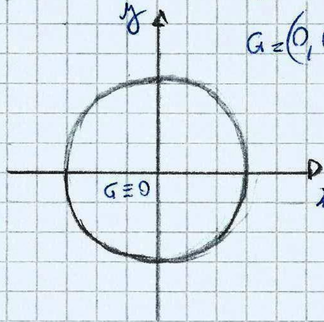
Quindi:

$$I_o = \begin{pmatrix} \frac{Mb^2}{3} & -M\frac{ab}{4} & 0 \\ -M\frac{ab}{4} & M\frac{a^2}{3} & 0 \\ 0 & 0 & \frac{M}{3}(a^2+b^2) \end{pmatrix}$$

Calcoliamo G:

$$I_G = \begin{pmatrix} \frac{Mb^2}{3} & -\frac{Mab}{4} & 0 \\ -\frac{Mab}{4} & \frac{Ma^2}{3} & 0 \\ 0 & 0 & \frac{M}{3}(a^2+b^2) \end{pmatrix} - M \begin{pmatrix} \frac{b^2}{4} & -\frac{ab}{4} & 0 \\ -\frac{ab}{4} & \frac{a^2}{4} & 0 \\ 0 & 0 & \frac{a^2+b^2}{4} \end{pmatrix} = \begin{pmatrix} \frac{Mb^2}{12} & 0 & 0 \\ 0 & \frac{Ma^2}{12} & 0 \\ 0 & 0 & \frac{M(a^2+b^2)}{12} \end{pmatrix}$$

- cerchio di raggio R:



$G=(0,0)$

Baricentro:

$$x_G = \frac{\rho}{M}\int_{-R}^{R} x\,dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy = 0$$

$$y_G = \frac{\rho}{M}\int_{-R}^{R} y\,dy \int_{\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dx = 0$$

Entrate della matrice d'inerzia applicando la trasformazione in coordinate polari:

$$\begin{cases} x = \varphi\cos\theta \\ y = \varphi\,\mathrm{sen}\,\theta \end{cases} \qquad \theta \in [0,2\pi],\ \varphi \in [0,R]$$

Quindi:

$$(I_0)_{11} = \iint \rho y^2\,dx\,dy = \rho\int_0^{2\pi} d\theta \int_0^R \varphi^2\,\mathrm{sen}^2\theta\cdot\varphi\,d\varphi =$$

$$= \frac{M}{\pi R^2}\int_0^{2\pi}\left[\frac{\varphi^4}{4}\right]_0^R\cdot\frac{1-\cos 2\theta}{2}\,d\theta = \frac{M}{\pi R^2}\cdot\frac{R^4}{4}\left[\frac{1}{2}\theta - \frac{\mathrm{sen}\,2\theta}{4}\right]_0^{2\pi} =$$

$$= \frac{MR^2}{\cancel{\pi}4}\cdot\cancel{\pi} = \frac{MR^2}{4}$$

Analogamente $(I_0)_{22} = \frac{MR^2}{4}$:

$$(I_0)_{22} = \iint \rho x^2\,dx\,dy = \frac{M}{\pi R^2}\int_0^{2\pi}\left[\frac{\varphi^4}{4}\right]_0^R\cdot\cos^2\theta\,d\theta = \frac{MR^2}{\pi 4}\cdot\left[\frac{1}{2}\theta + \frac{\mathrm{sen}\,2\theta}{4}\right]_0^{2\pi} =$$

$$= \frac{MR^2}{4}$$

Inoltre:

$$(I_0)_{1,2} = \iint \rho\,xy\,dx\,dy = \frac{M}{\pi R^2}\int_0^{2\pi} d\theta \int_0^R \varphi^3\cos\theta\,\mathrm{sen}\,\theta\,d\varphi =$$

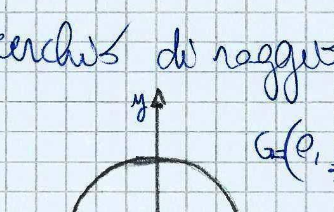
$$= \frac{M}{\pi R^2}\left[\frac{\varphi^4}{4}\right]_0^R\left[\frac{\mathrm{sen}^2\theta}{2}\right]_0^{2\pi} = 0$$

$$(I_0)_{33} = (I_0)_{11} + (I_0)_{22} = \frac{MR^2}{2}$$

Allora:

$$I_0 = I_G = \begin{pmatrix} \frac{MR^2}{4} & 0 & 0 \\ 0 & \frac{MR^2}{4} & 0 \\ 0 & 0 & \frac{MR^2}{2} \end{pmatrix}$$

- semicerchio di raggio R:



$G=\left(0, \frac{4}{3\pi}R\right)$

Baricentro:

$$x_G = \frac{\rho}{M}\int_{-R}^{R} x\,dx \int_0^{\sqrt{R^2-x^2}} dy = \frac{\rho}{M}\int_{-R}^{R}\sqrt{R^2-x^2}\cdot x\,dx =$$

$$= -\frac{\rho}{M}\cdot\frac{2}{3}\cdot\frac{1}{2}\left[(R^2-x^2)^{\frac{3}{2}}\right]_{-R}^{R} = 0$$

La coordinata y_G non è nulla!

$$y_G = \frac{\rho}{M}\iint y\,dx\,dy = \frac{2}{\pi R^2}\int_{-R}^{R} dx \int_0^{\sqrt{R^2-x^2}} y\,dy = \frac{2}{\pi R^2}\int_{-R}^{R}\frac{(R^2-x^2)}{2}dx =$$

$$= \frac{1}{\pi R^2}\left[R^2 x - \frac{x^3}{3}\right]_{-R}^{R} = \frac{1}{\pi R^2}\left(R^3 - \frac{R^3}{3} + R^3 - \frac{R^3}{3}\right) =$$

$$= \frac{1}{\cancel{\pi R^2}}\,\frac{4}{3}\,\cancel{R^3} = \frac{4}{3\pi}R$$

Entrate della matrice d'inerzia con la trasformazione in coordinate polari:

$$\begin{cases} x = \varphi\cos\theta \\ y = \varphi\,\mathrm{sen}\,\theta \end{cases} \qquad \theta\in[0,\pi] \quad \varphi\in[0,R]$$

Quindi:

$$(I_0)_{11} = \rho\iint y^2\,dx\,dy = \rho\int_0^{\pi}d\theta\int_0^R \varphi^3\,\mathrm{sen}^2\theta\,d\varphi =$$

$$= \frac{2M}{\pi R^2}\cdot\left[\frac{\varphi^4}{4}\right]_0^R\cdot\int_0^{\pi}\frac{1-\cos 2\theta}{2}\,d\theta = \frac{2M}{4\pi}R^2\cdot\left[\frac{1}{2}\theta - \frac{\mathrm{sen}\,2\theta}{4}\right]_0^{\pi} =$$

$$= \frac{2M}{\cancel{\pi}4}R^2\cdot\frac{1\cancel{\pi}}{2} = \frac{MR^2}{4}$$

$$(I_0)_{22} = \rho\iint x^2\,dx\,dy = \rho\int_0^{\pi}d\theta\int_0^R\varphi^3\cos^3\theta\,d\varphi =$$

$$= \frac{2M}{\pi R^2}\left[\frac{\varphi^4}{4}\right]_0^R\cdot\int_0^{\pi}\frac{1+\cos 2\theta}{2}\,d\theta = \frac{2M}{\pi 4}R^2\left[\frac{1}{2}\theta + \frac{\mathrm{sen}\,2\theta}{4}\right]_0^{\pi} =$$

$$= \frac{2M}{\cancel{\pi}4}R^2\cdot\frac{1}{\cancel{2}}\cancel{\pi} = \frac{MR^2}{4}$$

$$(I_0)_{12} = \rho\iint xy\,dx\,dy = \frac{2M}{\pi R^2}\int_0^{\pi}d\theta\int_0^R\varphi^3\cos\theta\,\mathrm{sen}\,\theta\,d\varphi =$$

$$= \frac{2M}{\pi R^2}\left[\frac{\varphi^4}{4}\right]_0^R\cdot\left[\frac{-\cos^2\theta}{2}\right]_0^{\pi} = 0$$

$$(I_0)_{33} = (I_0)_{11} + (I_0)_{22} = \frac{MR^2}{2}$$

Allora:

$$I_0 = \begin{pmatrix} \frac{MR^2}{4} & 0 & 0 \\ 0 & \frac{MR^2}{4} & 0 \\ 0 & 0 & \frac{MR^2}{2}\end{pmatrix}$$

Calcoliamo I_G:

$$I_G = \begin{pmatrix} \frac{MR^2}{4} & 0 & 0 \\ 0 & \frac{MR^2}{4} & 0 \\ 0 & 0 & \frac{MR^2}{2}\end{pmatrix} - M\begin{pmatrix} \frac{16}{9\pi^2}R^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{16}{9\pi^2}R^2\end{pmatrix} = MR^2\begin{pmatrix} \frac{9\pi^2-64}{36\pi^2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{8\pi^2-32}{18\pi^2}\end{pmatrix}$$